Skipping Ropes: An Efficient Gray Code Algorithm for Generating Wiggly Permutations

Original Paper by Vincent Pilaud and Aaron Williams

Atishaya Maharjan

GADA Lab

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Binary Reflected Gray Codes

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A ***Binary Reflected Gray Code*** is the sequence of binary numbers that differ by one bit.

Decimal	3-bit Binary Reflected Gray Code
0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

Binary Reflected Gray Codes

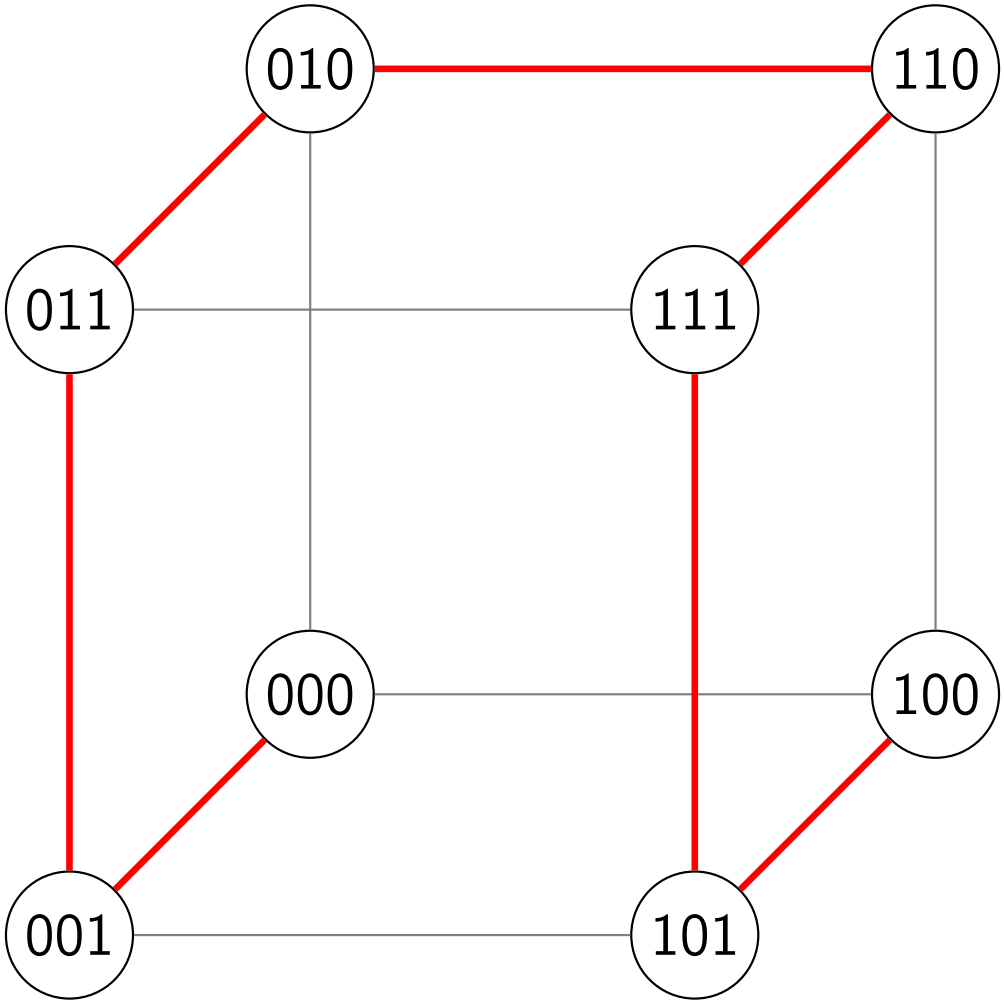

Figure: A Hamiltonian path on Q_3 is equivalent to listing the Gray codes in sequence.

Generation of Binary Reflected Gray Codes

We can formulate the Binary Reflected Gray Codes in a recursive manner:

$$B(n) = \begin{cases} 0 \cdot B(n-1),\ 1 \cdot B^R(n-1), & \text{for } n > 1 \\ 0, 1, & \text{for } n = 1 \end{cases}$$

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So for example we would have:

$$B(1) = 0, 1$$
$$B(2) = 00, 01, 11, 10$$
$$B(3) = 000, 001, 011, 010, 110, 111, 101, 100$$

Beyond Binary Strings: Combinatorial Objects

Natural Question: Can we create Gray codes for other combinatorial families?

Examples of Combinatorial Objects:

- **Combinations:** All k-element subsets of $\{1, 2, \ldots, n\}$
- **Permutations:** All orderings of $\{1, 2, \ldots, n\}$
- **Partitions:** All ways to partition a set into blocks
- **Trees:** All spanning trees of a graph
- **Strings:** Words with various constraints

In this talk, we will mostly consider permutations.

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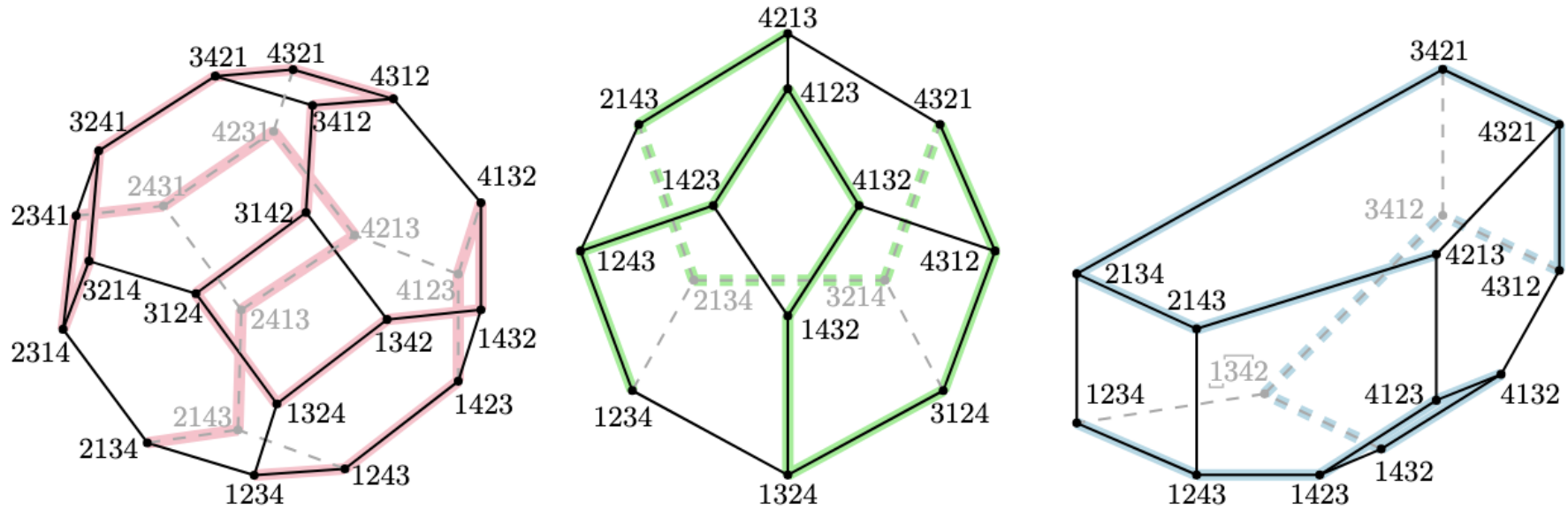
- Plain changes was discovered independently multiple times by a lot of different people, and the pattern became known as the Steinhaus-Johnson-Trotter algorithm [3].
- We can visualize this problem using flip graphs!

Flip graphs and plain changes



(a) Permutahedron for $n = 4$. **(b)** Associahedron for $n = 4$. **(c)** Wigglyhedron for $n = 4$.

Figure: Flip graph visualizations using different schemes [2]

Beyond Binary Strings: Combinatorial Objects

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Redefining Gray Codes

A gray code is simply the order of successive objects on the path of a flip graph.

- There has been a lot of interest in finding out simple and efficient generating algorithms for gray codes for various permutation objects.

Johnson-Trotter Algorithm for All Permutations

Goal: Generate all $n!$ permutations using only adjacent transpositions.

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Algorithm:

1. Start with $\pi = 12 \cdots n$
2. Each element has a direction (left $\leftarrow$ or right $\rightarrow$)
3. An element is *mobile* if it's larger than its neighbor in its direction
4. Repeatedly:
   - Find largest mobile element m
   - Swap m with neighbor in its direction
   - Reverse direction of all elements $> m$

Result: Generates all permutations, each differing from previous by one adjacent swap.

Johnson-Trotter Example: $n = 3$

Step	Permutation	Action
0	$\overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{3}$	largest mobile is 3
1	$\overleftarrow{1}\ \overleftarrow{3}\ \overleftarrow{2}$	3 moved left, largest mobile is 3
2	$\overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{2}$	3 moved left, reverse dir of none, largest mobile is 2
3	$\overleftarrow{3}\ \overleftarrow{2}\ \overrightarrow{1}$	2 moved left, reverse dir of 3, largest mobile is 3
4	$\overleftarrow{2}\ \overleftarrow{3}\ \overrightarrow{1}$	3 moved left, largest mobile is 3
5	$\overleftarrow{2}\ \overrightarrow{1}\ \overrightarrow{3}$	3 moved left, no mobile

Sequence: $123 \to 132 \to 312 \to 321 \to 231 \to 213$

Permutations: Basic Definitions

Definition

Permutation A permutation of $[n] = \{1, 2, \ldots, n\}$ is a bijection $\pi : [n] \to [n]$.

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$$\pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 4, \quad \pi(4) = 2$$

Inverse: $\pi^{-1}(j)$ is the position where value j appears.

For $\pi = 3142$: $\pi^{-1}(1) = 2$, $\pi^{-1}(2) = 4$, $\pi^{-1}(3) = 1$, $\pi^{-1}(4) = 3$

Inversions

Definition (Inversion)

For a permutation π of $[n]$, an *inversion* is a pair (i, j) with $i < j$ but $\pi(i) > \pi(j)$. The *inversion set* is $\mathrm{inv}(\pi)$.

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Example. $\pi = 3\ 1\ 4\ 2$.

Pairs (positions): $(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$

Inversions: $(1,2)$ since $3 > 1$; $(1,4)$ since $3 > 2$;

$(3,4)$ since $4 > 2$.

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Remark: inversion count measures distance from the identity.

Cover Relation

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σ *covers* π if $\pi < \sigma$ and there is no τ with $\pi < \tau < \sigma$. In our context this means $\mathrm{inv}(\sigma) = \mathrm{inv}(\pi) \cup \{(u, v)\}$ — exactly one new inversion is added.

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Example (n=3):

$$123 \quad \longrightarrow \quad 132$$

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Takeaway: In our case, cover edges correspond to adding a single inversion.

Edge labeling for the greedy rule

Label of a cover edge (π, σ). For a cover (one inversion added) the edge is labeled with the *larger value* of the new inversion. Equivalently: if (u, v) is the new inversion with $u < v$, label the edge by v.

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Why this label? The greedy algorithm picks, at each step, an available cover edge with *largest* label. Intuition: "move the largest value first" (analogue of Johnson–Trotter).

Worked example (from the paper):

$$\pi = 1\ 3\ 4\ 2 \quad \longrightarrow \quad \sigma = 1\ 4\ 3\ 2$$

- $\mathrm{inv}(\pi) = \{(3,2),(4,2)\}$, $\mathrm{inv}(\sigma) = \{(3,2),(4,2),(4,3)\}$.
- New inversion: $(4,3)$ so the edge (π, σ) gets label 4.

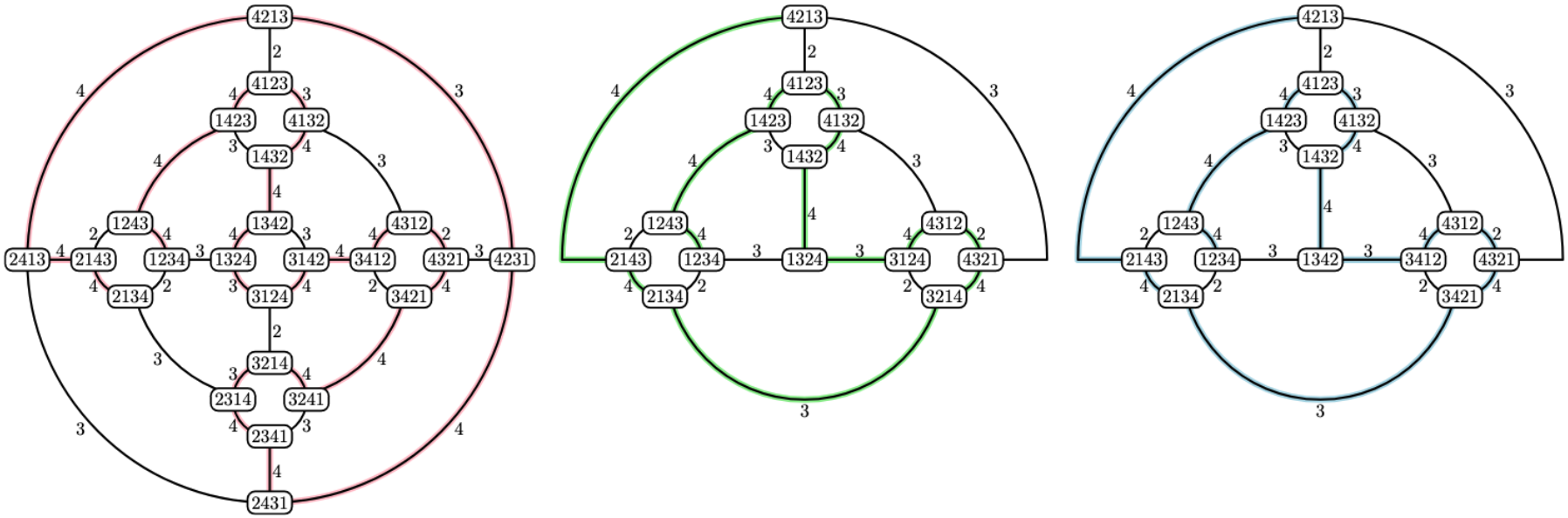
S_3 tiny table: covers and labels

Permutations of 3 and their cover-edges (weak order covers).

perm π	covers σ	edge label(s)
123	132	3
123	213	2
132	312	3
132	231	2
213	231	3
213	312	2

Greedy walk idea: from a node choose outgoing cover with largest label (e.g. from 123 choose label 3-edge $123 \to 132$).

Edge labelled algorithm visualization



(a) S_4 with swaps. **(b)** $\mathsf{Av}_4(231)$ with minimal jumps. **(c)** W_4 with wiggly flips.

Local Moves for Permutations

Common local moves:

1. **Adjacent transposition:** Swap π_i and π_{i+1}

$$\pi = \cdots \pi_i \, \pi_{i+1} \cdots \;\rightarrow\; \cdots \pi_{i+1} \, \pi_i \cdots$$

Example: $3142 \rightarrow 1342$ (swap positions 1 and 2)

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   Example: $3142 \rightarrow 1342$ (swap positions 1 and 2)
2. **Arbitrary transposition:** Swap any two elements
3. **Star transposition:** Swap π_1 with π_i for any i
4. **Reversal:** Reverse a contiguous subsequence

Note: Adjacent transpositions are most restrictive, hence most interesting for Gray codes!

Pattern-Avoiding Permutations

Definition. A permutation π *avoids* a pattern σ if π contains no subsequence order-isomorphic to σ.

Example: $\pi = 2413$ avoids 123 because no three positions have increasing values.

Classical results:

- Permutations avoiding 123 are counted by Catalan numbers
- Many pattern classes have elegant enumeration formulas
- Rich connection to algebraic combinatorics

Definition: Wiggly Permutations

Definition [1] A permutation $\pi \in S_n$ is *wiggly* if for all odd j with $1 \leq 2j \leq n$:

$$\begin{aligned} &\textbf{If } 2j-1 \text{ appears } \textbf{left} \text{ of } 2j \text{ in } \pi: && \pi^{-1}(2j-1) < \pi^{-1}(2j) \\ &\textbf{If } 2j-1 \text{ appears } \textbf{right} \text{ of } 2j \text{ in } \pi: && \pi^{-1}(2j-1) > \pi^{-1}(2j) \end{aligned}$$

Intuition: For each pair $(2j-1, 2j)$, the *relative order* in the permutation must match the *relative position*.

Notation: W_n = set of all wiggly permutations of $[n]$.

Example: Verifying $\pi = 2413$ is Wiggly

Given: $\pi = 2413$ (so $\pi_1 = 2, \pi_2 = 4, \pi_3 = 1, \pi_4 = 3$)

Inverse function:

- $\pi^{-1}(1) = 3$ (value 1 is at position 3)
- $\pi^{-1}(2) = 1$ (value 2 is at position 1)
- $\pi^{-1}(3) = 4$ (value 3 is at position 4)
- $\pi^{-1}(4) = 2$ (value 4 is at position 2)

Check pair $(1, 2)$ (where $j = 1$):

- Value 1 appears at position 3, value 2 at position 1
- So 2 appears *left* of 1 in the permutation
- Need: $\pi^{-1}(1) > \pi^{-1}(2)$ (right appears right)
- Check: $3 > 1$

Check pair $(3, 4)$ (where $j = 2$):

- Value 3 appears at position 4, value 4 at position 2
- So 4 appears *left* of 3 in the permutation
- Need: $\pi^{-1}(3) > \pi^{-1}(4)$ (right appears right)
- Check: $4 > 2$

Small Examples of Wiggly Permutations

n	W_n (all wiggly permutations)
1	1
2	12
3	132, 231
4	1324, 2143, 2413, 3142, 3412

Observation: $|W_n|$ grows much slower than $n!$

- $|W_1| = 1,\ |W_2| = 1,\ |W_3| = 2,\ |W_4| = 5,\ |W_5| = 16,\ |W_6| = 61$
- **Note:** The full sequence remains unknown and there is no work (Per my knowledge) of a closed form solution for $|W_n|$ as n grows.

Blocks, Jumps and Hops

Block. A block is a contiguous subsequence of a permutation.

Moves used in wiggly flips:

- **Jump:** move a block of length 1 (a single element) past an adjacent block.
- **Hop:** move a block of length 2 (two adjacent elements) past an adjacent block.

Example (n=4):

$$\pi = 1\ 3\ 4\ 2$$

- Jump: move element 4 left past 3 $\Rightarrow$ 1 4 3 2.
- Hop: move block [3 4] left past 1 $\Rightarrow$ 3 4 1 2.

Remark: hops/jumps are minimal legal moves that preserve the wiggly constraints.

Problem Statement

Question: Does there exist a Gray code for W_n using wiggly flips (jumps and hops)?

More precisely: Can we list all wiggly permutations such that:

1. Each permutation appears exactly once
2. Consecutive permutations differ by a single legal wiggly flip (jump or hop)
3. The flip preserves the wiggly property

Equivalently: Does the flip graph of W_n (with edges for legal wiggly flips) have a Hamiltonian path?

Main Theorem

Theorem [2]. For all $n \geq 1$, there exists a Gray code of wiggly permutations (a Hamilton path in the wiggly flip graph).

Moreover, there is an algorithm that:

- Generates all wiggly permutations in Gray code order.
- Uses $\mathcal{O}(n)$ space.
- Runs in worst case $\mathcal{O}(n)$ time per permutation (Conjectured to be $\mathcal{O}(1)$ amortized).
- Is constructive and can be implemented efficiently.

Algorithm F — Greedy edge-label walk (conceptual)

Purpose: Produce a Hamilton path of the wiggly flip graph (existence proof). Algorithm 2 will actually implement this.

Key ingredients:

- Label each cover edge (flip) by the *larger value* involved in the new inversion added.
- From the current permutation, consider all legal wiggly flips (jumps/hops) and their labels.
- **Greedy rule:** always perform an available flip of *maximum* label (deterministic tie-break).

Outcome: The greedy walk (Algorithm F) visits every wiggly permutation exactly once (produces a Hamilton path). (paper: existence theorem)

Short intuition: moving larger values first creates a structured sweep (like Johnson–Trotter) and enforces global invariants that prevent revisiting vertices.

Algorithm 2 / WiggleGray — implementation (loopless style)

Goal: Implement the Algorithm F order as a generator with provable resource bounds.

High level:

- Use the recursive insertion-based strategy (build W_n from W_{n-1}).
- Maintain the set of currently-available flips and their labels incrementally (no full scan).
- At each step pick the maximal-label flip, perform it, update only local candidates.

For theoretical bound guarantees: Refer to the paper.

Algorithm Pseudocode

WiggleGray(n):

1. **Base case:** If $n = 1$, output (1) and return
2. Let $L \leftarrow$ WiggleGray($n - 1$) (recursive call)
3. Initialize direction dir $\leftarrow$ forward
4. **For each** $\pi \in L$ (traversed according to dir):
   1. Compute set P of valid positions for inserting n into π
   2. Order positions in P according to dir
   3. **For each** position $p \in P$:
      - Output: π with n inserted at position p
   4. Toggle: dir $\leftarrow$ (forward $\leftrightarrow$ backward)

Key: Alternating directions ensures outputs connect via single adjacent swaps / wiggly flips.

Example: W_3 Gray Code

Start: $W_2 = \{12\}$

Build W_3 by inserting 3:

- Base permutation: $\pi' = 12$
- Valid positions: need to check wiggly constraints for pair $(2, 3)$
- Try position 2: 132
  - $\pi^{-1}(2) = 3, \pi^{-1}(3) = 2$
- Try position 3: 123
  - $\pi^{-1}(2) = 2, \pi^{-1}(3) = 3$

perm	inv	word	fs	dirs
1234	1234	0000	1234	−−−−
1243	1243	0001	1234	−−−−
1423	1342	0002	1234	−−−−
4123	2341	0003	1233	−−−+
4132	2431	0013	1234	−−−+
1432	1432	0012	1234	−−−+
1342	1423	0011	1233	−−−−
3412	3412	0022	1224	−−+−
4312	3421	0023	1232	−−++
4321	4321	0123	1134	−+++
3421	4312	0122	1133	−+−−
2134	2134	0100	1214	−+−−
2143	2143	0101	1214	−+−−
4213	3241	0103	1231	−+−+

Figure: Wiggly changes of 1234 with the Algorithm visualized [2]

Future Directions

- Tighter analysis of the algorithm will probably yield better average or amortized results.

[1] Asilata Bapat and Vincent Pilaud. “Wigglyhedra”. In: *Mathematische Zeitschrift* 310.3 (2025), p. 54.

[2] Vincent Pilaud and Aaron Williams. “Skipping Ropes: An Efficient Gray Code Algorithm for Generating Wiggly Permutations”. In: *19th International Symposium on Algorithms and Data Structures (WADS 2025)*. Ed. by Pat Morin and Eunjin Oh. Vol. 349. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025, 46:1–46:20. ISBN: 978-3-95977-398-0. DOI: 10.4230/LIPIcs.WADS.2025.46. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.WADS.2025.46.

[3] Hugo Steinhaus. *One Hundred Problems in Elementary Mathematics*. New York, NY: Courier Corporation, 1979.