Exploring perfect binary trees with relation to the HK-property

MXML Presentation

Atishaya Maharjan

April 3, 2024

Outline

EKR Theorem

The **Erdős-Ko-Rado** theorem limits the number of sets in an intersecting family.

Theorem (EKR Theorem)

[a] *If $\mathcal{F}$ is an intersecting family of k-subsets of an n-set (cardinality of the set is n), then*

- $|\mathcal{F}| \leq \binom{n-1}{k-1}$
- *If equality holds, $\mathcal{F}$ consists of the k-subsets that contain i, for some i in the n-set.*

---

[a] Erdös, Ko, and Rado, "INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS".

HK-property

Definition (Cocliques)

- A **coclique** in a graph is a set of vertices such that no two vertices in the set are adjacent.
- The maximum size of a coclique in a graph is called the **maximum coclique** of the graph. For a graph G, it is denoted by $\alpha(G)$.

HK-property

Definition (Cocliques)

- A **coclique** in a graph is a set of vertices such that no two vertices in the set are adjacent.
- The maximum size of a coclique in a graph is called the **maximum coclique** of the graph. For a graph G, it is denoted by $\alpha(G)$.

Definition (Stars and Stars Center)

- In a graph G, the set of all cocliques of a fixed size k, including a fixed vertex v is called a star centered at v.
- It is denoted by $\mathcal{I}_G^k(v)$.

HK-property

Definition (Cocliques)

- A **coclique** in a graph is a set of vertices such that no two vertices in the set are adjacent.
- The maximum size of a coclique in a graph is called the **maximum coclique** of the graph. For a graph G, it is denoted by $\alpha(G)$.

Definition (Stars and Stars Center)

- In a graph G, the set of all cocliques of a fixed size k, including a fixed vertex v is called a star centered at v.
- It is denoted by $\mathcal{I}_G^k(v)$.

Definition (k-EKR graph)

Let $\mathcal{F}$ be a family of cocliques of fixed size k in a graph G such that any two sets in $\mathcal{F}$ have a non-empty intersection. Then, there exists a vertex v such that $|\mathcal{F}| \leq \mathcal{I}_G^k(v)$.

HK-property

Studying the EKR theorem,[1] Hurlbert and Kamat made the following conjecture:

Conjecture (HK-Property)

For any $k \geq 1$ and any tree T, there exists a leaf l of T such that $|\mathcal{I}_T^k(v)| \leq |\mathcal{I}_T^k(l)|$ for each $v \in V(T)$.

[1] Hurlbert and Kamat, "Erdős-Ko-Rado theorems for chordal graphs and trees".

HK-property

- The HK-property was proven for $k \leq 4$, but the conjecture was shown to be false.[2][3][4]

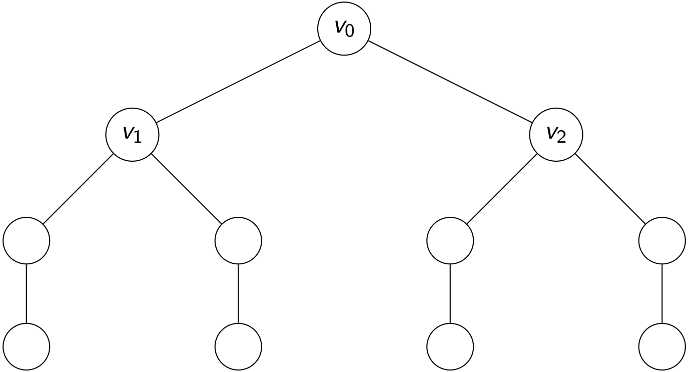


Figure 1: The largest k-star for $k \geq 5$ is centered at v_0

[2] Borg and Holroyd, "The Erdős-Ko-Rado properties of various graphs containing singletons".
[3] Borg, "Stars on trees".
[4] Baber, *Some results in extremal combinatorics*.

Some graphs that DO satisfy the HK-property[5]

Caterpillars:

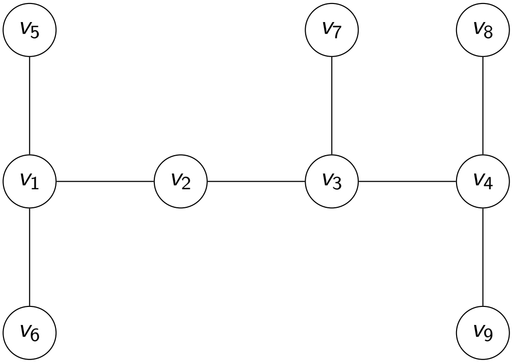


Figure: A caterpillar

[5] Hurlbert and Kamat, "Erdős-Ko-Rado theorems for chordal graphs and trees".

Some graphs that DO satisfy the HK-property[6]

Spiders:

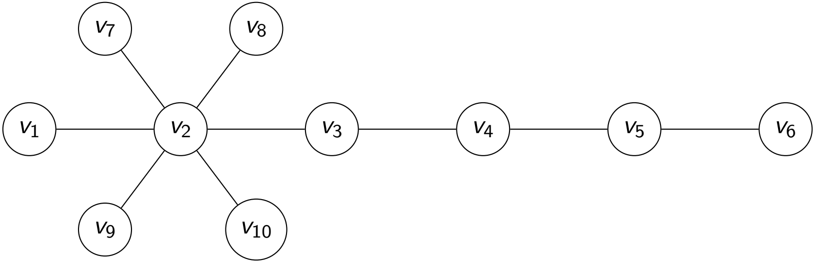


Figure: A Spider

[6] Hurlbert and Kamat, "Erdős-Ko-Rado theorems for chordal graphs and trees".

Some graphs that DO satisfy the HK-property[7]

Lobsters*:

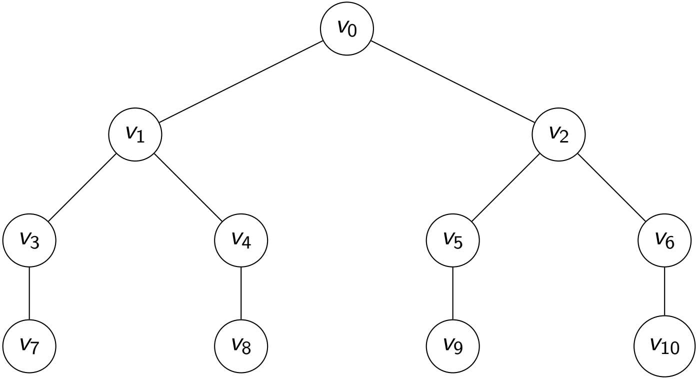
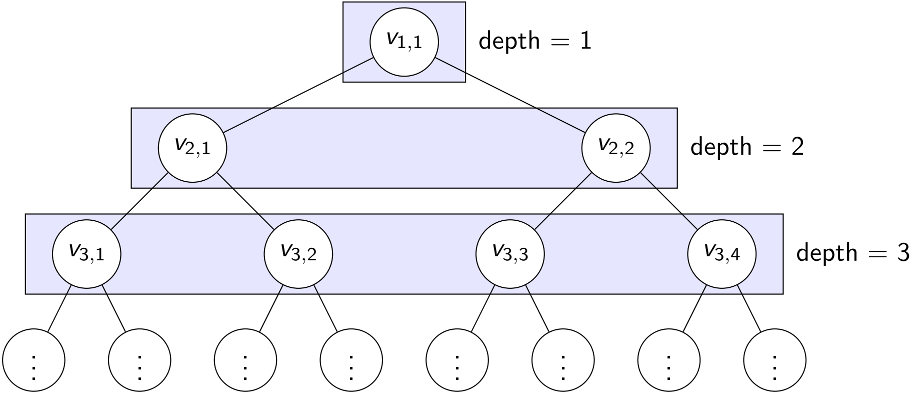


Figure: A Lobster

[7] Hurlbert and Kamat, "Erdős-Ko-Rado theorems for chordal graphs and trees".

Perfect Binary Tree

Perfect Binary Tree:



Does a perfect binary tree satisfy the HK property?

1. At least partially.

Does a perfect binary tree satisfy the HK property?

1. At least partially.
2. The lobster almost satisfies the HK property and the perfect binary tree has a close relation to the lobster.

Does a perfect binary tree satisfy the HK property?

1. At least partially.
2. The lobster almost satisfies the HK property and the perfect binary tree has a close relation to the lobster.
3. In addition, the perfect binary tree is very symmetric and has a lot of structure that we can maniputlate.

Using an enumeration approach with computer algorithms to verify the results

Before we proceed with proving anything, it would be helpful to first verify some results and get some data using computer algorithms.

Using an enumeration approach with computer algorithms to verify the results

Data: A perfect binary tree graph T
Result: All cocliques of T

```
Function enumerate_cocliques(T):
    cocliques ← [];
    cocliques.append(∅);
    for vertex in T do
        new_cocliques ← [];
        for coclique in cocliques do
            for neighbor in vertex.neighbors do
                if neighbor ∉ coclique then
                    new_coclique ← coclique ∪ {neighbor};
                    new_cocliques.append(new_coclique);
                end
            end
        end
        cocliques ← new_cocliques;
    end
    return cocliques
```

Using an enumeration approach with computer algorithms to verify the results

1. The results do indeed verify that the HK-property holds for perfect binary trees of depth 5. With the pattern, it might hold for any depth perfect binary tree.

Using an enumeration approach with computer algorithms to verify the results

1. The results do indeed verify that the HK-property holds for perfect binary trees of depth 5. With the pattern, it might hold for any depth perfect binary tree.
2. It does also show us that all the leaves are included in the maximum coclique.

Using an enumeration approach with computer algorithms to verify the results

1. The results do indeed verify that the HK-property holds for perfect binary trees of depth 5. With the pattern, it might hold for any depth perfect binary tree.
2. It does also show us that all the leaves are included in the maximum coclique.
3. However, patterns have a history of misleading mathematicians and as such, a proof is needed.

Inductive Approach

We can conjecture a formula for the maximum coclique of a perfect binary tree of depth d:

Conjecture

For any perfect binary tree T of depth d, the maximum coclique $\alpha(T)$ is given by

$$\alpha(T) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} 2^{d-2i}$$

Furthermore, the maximum coclique is unique.

This is still a work in progress, but we believe that we can give an inductive proof for this conjecture by inducting on d.

Inductive Approach

If the previous conjecture holds, then we claim that:

Claim

There is a unique maximum coclique set that contains all the leaves.

Then from the claim and all the observations, we can conjecture the following:

Conjecture

Let T be a perfect binary tree of depth d. Let r be the root of T. Then, for all possible values of d and k, there exists a leaf l of T such that $|\mathcal{I}_T^k(v)| \leq |\mathcal{I}_T^k(l)|$ for each $v \in V \setminus r$.

Which is partially the HK conjecture. Note that this is the exact statement for the HK conjecture for lobsters given by[8] Estrugo and Pastine.

[8] Estrugo and Pastine, "On stars in caterpillars and lobsters".

Open Questions and Future work

Note that a perfect binary tree is just a specific case for a perfect k-nary tree. So, we can most likely generalize the results for perfect k-nary trees to show that they satisfy a partial HK-property, similar to those of the binary tree.

In addition, we can also investigate the HK-property for other types of binary trees (Full, Complete, Normal, etc) and see if they satisfy the HK-property.

Thank You!

Thank you for listening!